Indian J. Phys. **44**, 386–391, (1970)

Nonlinear interaction of electromagnetic waves in a magnetized electron beam

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(*Received* 7 *November* 1969—*Revised* 21 *March and* 3 *August* 1970)

The nonlinear interaction of electromagnetic waves in a magnetized electron beam has been investigated. The waves are taken to be propagating across a uniform magnetic field. It has been established here that extraordinary waves are generated when two ordinary and two extraordinary waves interact

INTRODUCTION

The phenomenon related to the nonlinear interaction of electromagnetic waves in a plasma plays a basic role (Kroll *et al* 1964, Montgomery 1965) in the study of weak plasma turbulence. The study of coherent interaction of two electromagnetic waves in the ionosphere is an example of the application of this phenomenon in which Blachier & Bouchet (1966) have proposed a scheme of radio communication over a long distance. Etievant *et al* (1968) have discussed the nonlinear interaction of electromagnetic waves in a cold magnetised plasma and shown that (a) interaction of an extraordinary wave with an ordinary wave generates an ordinary wave, and (b) interaction of two ordinary waves generates an extraordinary wave.

In a hot plasma several interaction mechanisms (Clauser 1960) leading to the generation as well as amplification of microwave signals have been suggested. The hot plasma is regarded as of the form of an electron beam whose directed velocity is much greater than the random velocity of the electrons. We have considered the nonlinear interaction of electromagnetic waves propagating across a uniform magnetic field in an electron beam. The nonlinear effects are treated by considering the waves which arise in the linear theory and interact with each other It is found that for an appropriate choice of these electromagnetic waves a secondary wave is generated

FIRST ORDER PERTURBATIONS

In equilibrium, an isotropic plasma consisting of electrons and subjected to a constant uniform magnetic field $\vec{B} = B\hat{z}$, may be regarded as a single beam with no perpendicular energy and with a constant zero-order parallel velocity $\vec{V} = V\hat{x}$. The basic equations are similar to those of Etievant *et al* (1968). The equations

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governing the first order perturbations $\vec{v}_1$, $\vec{E}_1$, $\vec{h}_1$ and n_1 respectively in $\vec{V}$, $\vec{E}$, $\vec{B}$ and N come out to be

$$\left(\frac{\partial}{\partial t}+\vec{V}.\nabla\right)\vec{v}_1 = -\frac{e}{m}\left(\vec{E}_1+\frac{\vec{V}\times\vec{h}_1}{c}+\frac{\vec{v}_1\times\vec{B}}{c}\right), \quad \ldots (1)$$

$$\nabla\times\vec{E}_1 = -\frac{1}{c}\frac{\partial\vec{h}_1}{\partial t} \quad \ldots (2)$$

$$\nabla\vec{E}_1 = -4\pi n_1 e \quad \ldots (3)$$

$$\nabla\times\vec{h}_1 = \frac{1}{c}\frac{\partial\vec{E}_1}{\partial t}-\frac{4\pi e}{c}(n_1\vec{V}+N\vec{v}_1); \quad \ldots (4)$$

$$\nabla.\vec{h}_1 = 0.$$

In equations (1)—(5), we choose the time dependence of the first order perturbed quantities of the form $\exp(-i\omega t)$. On eliminating $\vec{h}_1$ between equations (1) and (2), we operate successively equation (1) by the operator $i\omega-\vec{V}\nabla$, and obtain

$$i\omega(i\omega-\vec{V}.\nabla)[(i\omega-\vec{V}\nabla)^2+\Omega^2]\vec{v}_1 = \frac{e}{m}\vec{\mu}.\vec{P}_1 \quad \ldots (6)$$

where

$$\vec{\mu} = \begin{bmatrix} (i\omega-\vec{V}.\nabla)^2 & \Omega(i\omega-\vec{V}.\nabla) & 0 \\ -\Omega(i\Omega-\vec{V}\nabla) & (i\omega-\vec{V}.\nabla)^2 & 0 \\ 0 & 0 & \Omega^2+(i\omega-\vec{V}.\nabla)^2 \end{bmatrix}, \quad \ldots (7)$$

and

$$\vec{P}_n = (i\omega-\vec{V}.\nabla+\nabla\vec{V}.)\vec{E}_n \quad \ldots (8)$$

Here $\Omega = eB/mc$ is the cyclotron frequency and in equation (8) the subscript n stands for the order of the perturbed quantity.

From equations (2)—(4), $\vec{v}_1$ when expressed in terms of $\vec{E}_1$, comes out to be of the form

$$\vec{v}_1 = \frac{e}{m}\frac{ic^2}{\omega\omega_0^2}\left[\left(\nabla-\frac{i\omega}{c^2}\vec{V}\right)\nabla.-\nabla^2-\frac{\omega^2}{c^2}\right]\vec{E}_1. \quad \ldots (9)$$

We eliminate v_1 between equations (6) and (9) and obtain the dispersion relation

$$(i\omega-\vec{V}.\nabla)[(i\omega-\vec{V}.\nabla)^2+\Omega^2]\left[(\nabla-\frac{i\omega}{c^2}\vec{V})\,\nabla.-\nabla^2-\frac{\omega^2}{c^2}\right]E_1$$

$$= -\frac{\omega_0^2}{c^2}\vec{\mu}.\vec{\mathbf{P}}_1 \quad \ldots (10)$$

If the variation of $\vec{E}_1$ is taken to be of the form $\vec{\epsilon} \exp i(kx-\omega t)$ then the dispersion relation (10) admits of two independent solutions

$$c^2k^2 = \omega^2-\omega_0^2, \qquad \text{ordinary waves,} \qquad \dots \quad (11)$$

$$c^2k^2 = \omega^2-\omega_0^2-\frac{\Omega^2\omega^2{}_0}{\alpha^2-\Omega^2-\omega_0^2}, \text{ extraordinary waves,} \qquad \dots \quad (12)$$

where

$$\alpha = (\omega-kV), \ \omega_0^2 = 4\pi Ne^2/m. \qquad \dots \quad (13)$$

From equations (11) and (12), it is evident that two types of waves, ordinary and extraordinary, propagate in a magnetized electron beam.

For propagations perpendicular to the magnetic field the first order perturbations for the extraordinary waves are found to be

$$\vec{E}_e = A(\hat{x}-ia\hat{y}), \quad \vec{v}_e = -\frac{ic}{m}\frac{A}{\omega_0^2}\left[\alpha\hat{x}+\frac{i}{\Omega}(\alpha^2-\omega_0^2)\hat{y}\right],$$

$$\vec{h}_e = -\frac{iackA}{\omega}\hat{z}, \quad n_e = -\frac{ikA}{4\pi e}, \qquad \dots \quad (14)$$

where A is the normalization factor given by

$$A^2 = \frac{|\epsilon_e|^2}{1-a^2}, \quad a = \frac{\omega\Omega}{c^2k^2+\omega_0^2-\omega^2}; \qquad \dots \quad (14a)$$

and for the ordinary waves we have

$$\vec{E}_0 = \epsilon_{\|}\hat{z}\ \vec{V}_0 = -\frac{ie}{m\omega}\epsilon_{\|}\hat{z}, \quad \vec{h}_0 = -\frac{ck}{\omega}\epsilon_{\|}\hat{y}, \quad n_0 = 0. \qquad \dots \quad (15)$$

Here in equations (14) and 15), we have omitted the exponential factor $\exp i(kx-\omega t)$.

Second Order Perturbations

In a way similar to that adopted for the first order perturbations, we obtain the dispersion relation for the second order perturbations which yields

$$E_x^2 = -\frac{\alpha c^2}{\omega\delta}\left[i\{\alpha^2\omega_0^2+(\alpha^2-\Omega^2)(c^2k^2-\omega^2)\}J_{sx}-\alpha\Omega\omega_0^2J_{sy}\right], \qquad \dots \quad (16)$$

$$E_y^2 = -\frac{c^2}{\delta}[\Omega\omega_0^2(\Omega-kV)J_{sx}-i\alpha^2(\alpha^2-\Omega^2-\omega_0^2)J_{sy}], \qquad \dots \quad (17)$$

$$E_z^2 = \frac{ic^2J_{sz}}{\alpha(\alpha^2-\Omega^2)(c^2k^2-\omega^2+\omega_0^2)}, \qquad \dots \quad (18)$$

where

$$\vec{J}_s = \frac{\omega\omega_0^2}{c^2}\frac{m}{e}(i\alpha^2-\alpha\Omega\hat{z}\times-i\Omega^2\hat{z}\hat{z}_0)\left[(\vec{v}_1\cdot\nabla)\vec{v}_1+\frac{e}{mc}(\vec{v}_1\times\vec{h}_1)\right]$$

$$\frac{4\pi e a}{c^2}\quad\alpha(\alpha^2-\Omega^2)n_1 v_1, \qquad \dots (19)$$

$$\delta=\alpha^3(\alpha^2-\Omega^2-\omega_0^2)[\alpha^2\omega_0^2+(\alpha^2-\Omega^2)(c^2k^2-\omega^2)]+\alpha^2\Omega^2\omega_0^4(\Omega-kV). \qquad \dots(20)$$

Equations (16)—(18) determine the second order electric field, whereas, equation (19) determines the second order current source. The first order terms occurring in equation (19) are given by the set (14) for extraordinary waves and by the set (15) for the ordinary waves.

Interaction of Two Ordinary Waves

We choose the subscripts 1 and 2 to distinguish the two interacting ordinary waves and let $k_\pm=k_1\pm k_2$ etc.

From the set (15) it follows that the first order density for the ordinary waves is zero, therefore only the first term in equation (19) will contribute. In view of the fact that the first order velocity component is directed along the z axis, from equation (19) it is found that the interaction produces a current source given by

$$\vec{J}_s=\pm\frac{e}{m}\;\frac{\omega_0^2}{c^2}\;\frac{\epsilon_{\|1}\,\epsilon_{\|2}}{2\omega_1\omega_2}\alpha_\pm k_\pm\omega_\pm[\alpha_\pm\cos(k_\pm x-\omega_\pm t)\hat{x}$$

$$-\Omega\sin(k_\pm x-\omega_\pm t)\hat{y}]. \qquad \dots (21)$$

Equation (21) establishes that the current source is perpendicular to the magnetic field and therefore the second order wave is an extraordinary wave.

The second order electric field as calculated from equations (16)—(18) and (21), comes out to be

$$\vec{E}_2=\mp\frac{e}{m}\;\frac{\Omega}{\delta}\;\frac{\epsilon_{\|1}\,\epsilon_{\|2}}{2\omega_1\omega_2}\;\omega_0^2\alpha^2_\pm k_\pm\omega_\pm[\omega_0^2(\Omega-k_\pm V)+\alpha_\pm(\alpha^2_\pm-\Omega^2-\omega_0^2)]\hat{y}\,. \quad (22)$$

Interaction of Two Extraordinary Waves

The first order terms for the extraordinary waves are given by the set (14) which reveals that both the terms of equation (19) will contribute. The second order current source and electric field, calculated from equations (14) and (16)—(19), are given by

$$\vec{J}_2 = \pm \frac{e}{m} \frac{A_1 A_2}{2c^2} \alpha_\pm \omega_\pm [C_1 \cos(k_\pm x - \omega_\pm t)\hat{x} - C_2 \sin(k_\pm x - \omega_\pm t)\hat{y}], \quad \dots \quad (23)$$

$$\vec{E}_2 = \mp \frac{e}{m} \frac{A_1 A_2}{2\delta} \alpha_\pm \omega_\pm [\Omega \omega_0^2(\Omega - k_\pm V)C_1 + \alpha^2_\pm(\alpha^2_\pm - \Omega^2 - \omega_0^2)C_2]\hat{y}, \quad \dots \quad (24)$$

where

$$C_1 = \frac{\lambda_\pm}{\omega_0^2} \pm \frac{\Omega^2\phi + \alpha_\pm \psi_\pm}{\Omega\omega_1\omega_2},$$

$$C_2 = \frac{1}{\Omega\omega_0^2} [\alpha_1\alpha_2\Omega^2 k_\pm \pm (\alpha^2_\pm - \Omega^2)[k_1(\alpha_2^2 - \omega_0^2) \pm k_2(\alpha_1^2 - \omega_0^2)] + \alpha_\pm \chi]$$

$$+ \frac{\alpha_\pm \phi + \psi_\pm}{\omega_1\omega_2},$$

$$\begin{aligned}
\lambda_\pm &= \chi + \alpha_1\alpha_2\alpha_\pm k_\pm + (\alpha^2_\pm - \Omega^2)(\alpha_1 k_2 + \alpha_2 k_1), \\
\chi &= \alpha_2 k_1(\alpha_1^2 - \omega_0^2) + \alpha_1 k_2(\alpha_2^2 - \omega_0^2), \\
\phi &= \alpha_1\omega_1\alpha_2 k_2 + \alpha_2\omega_2 a_1 k_1 \\
\psi_\pm &= \pm[a_1 k_1 \omega_2(\alpha_2^2 - \omega_0^2) + a_2 k_2 \omega_1(\alpha_1^2 - \omega_0^2)] \quad \dots \quad (25)
\end{aligned}$$

The current source given by equation (23) is perpendicular to the magnetic field, therefore the interaction of two extraordinary waves generates an extraordinary wave Here the subscripts 1 and 2 distinguish the two interacting extraordinary waves. The quantities A and a are given by equation (14a) and δ is given by equation (20).

Interaction of an Ordinary Wave with an Extraordinary Wave

We distinguish the extraordinary wave and the ordinary wave by the subscripts 1 and 2 respectively. For the current source, only the second term of equations (19) contributes and it is found that

$$\vec{J}_s = \pm \frac{e}{m} \frac{A_1 k_1 \epsilon_{\parallel 2}}{2\omega_2 c^2} \alpha_\pm \omega_\pm (\alpha^2_\pm - \Omega^2) \cos(k_\pm x - \omega_\pm t)\hat{z}. \quad \dots \quad (26)$$

Equation (26) establishes that the interaction of an ordinary wave with an extraordinary wave generates an extraordinary wave.

In view of equation (11), equation (18) gives an indeterminate value of the second order electric field This conclusion shows that the consideration of the interaction of an extraordinary wave with an ordinary wave is not possible by this analysis.

ACKNOWLEDGEMENT

This work has been supported by the University of Jodhpur.

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